filter ($k=0$). The phase response becomes more nonlinear as k varies from 0 to 1.

Figs. 7 and 8 show the amplitude and phase responses of third-order TRXY filters varying from the elliptic filter to the Bessel rational filter of (7) with $m=2$, $n=3$, $\alpha=\beta=0$, and $\gamma=0.5$. The phase response improves as k varies from 0 to 1. These results are similar to those of Figs. 1 and 2. The Bessel rational filter was not frequency scaled in this case.

Transitional TRXY filters are important because they provide the filter designer a compromise between the desirable properties of the X filter and those of the Y filter. By proper choice of X and Y filters the designer is offered a wide selection of amplitude and phase responses.

References

[1] Y. Peless and T. Murakami, "Analysis and synthesis of transitional Butterworth-Thomson filters and bandpass amplifiers," *RCA Rev.*, vol. 18, pp. 60–94, 1957

[2] J. R. Johnson, D. E. Johnson, and R. J. LaCarna, "Transitional filters," *1978 IEEE Int. Symp. Circuits Syst. Proc.*, pp. 434–435, 1978.

[3] J. Attikiouzel and Dang Tan Phuc, "On transitional ultraspherical-ultraspherical filters," *Proc. IEEE*, vol. 66, pp. 703–706, June 1978.

[4] L. Weinberg, *Network Analysis and Synthesis.* New York: McGraw-Hill, 1962.

[5] D. E. Johnson, J. R. Johnson, and M. D. Kashefi, "Ultraspherical rational filters," *IEEE Trans. Circuit Theory*, vol. CT-20, pp. 596–599, Sept. 1973.

[6] J. R. Johnson, D. E. Johnson, P. W. Boudra, Jr., and V. P. Stokes, "Filters using Bessel-type polynomials," *IEEE Trans. Circuits Syst.*, vol. CAS-23, pp. 96–99, Feb. 1976.

[7] A. Budak, "A maximally flat phase and controllable magnitude approximation," *IEEE Trans. Circuit Theory*, vol. CT-12, p. 279, June 1965.

[8] D. E. Johnson, J. R. Johnson, and A. Eskandar, "A modification of the Bessel filter," *IEEE Trans. Circuits Syst.*, vol. CAS-22, pp. 645–648, Aug. 1975.

[9] J. Neirynck, "The attenuation phase compromise in the polynomial case," *Proc. 1972 NATO Advanced Study Inst. Network and Signal Theory*, 1973.

[10] J.R. Martinez, "Transfer functions of generalized Bessel polynomials," *IEEE Trans. Circuits Syst.*, vol. CAS-24, pp. 325–328, June 1977.

[11] B. D. Rakovich, M. V. Popovich, and B. S. Drakulich, "Minimum phase transfer functions providing a compromise between phase and amplitude approximation," *IEEE Trans. Circuits Syst.*, vol. CAS-24, pp. 718–724, Dec. 1977.

A New Algorithm for Computing a Single Root of a Real Continuous Function

C. J. F. RIDDERS

***Abstract*—A fast and simple iterative method is proposed for the determination of a single real root of a real continuous function. The idea is based upon linearizing the original function whereafter the *regula falsi* is applied to this modified function which leads to a very simple algorithm. The rate of convergence is shown to be quadratic or better.**

I. Method

Let the function be represented by $F(x)$. We create a new function $H(x)=F(x)\cdot e^{mx}$ in such a way that for three equidistant x values x_0, x_1 an x_2 the following requirement is met:

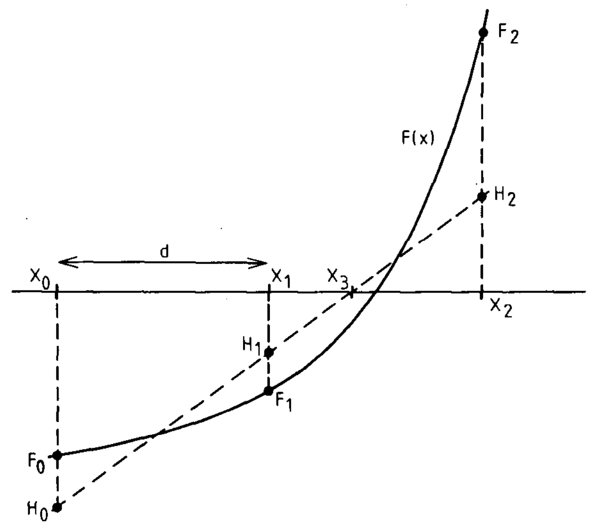

Fig. 1.

$$H_2-2H_1+H_0=0, \quad \text{with} \quad H_n=H(x_n). \tag{1}$$

Let $d=x_2-x_1=x_1-x_0$ and $F_0\cdot F_2<0$, then from (1) it follows

$$F_2\cdot e^{2md}-2F_1\cdot e^{md}+F_0=0 \tag{2}$$

with the analytical solution

$$e^{md}=\frac{F_1-\text{sign}(F_0)\cdot\sqrt{W}}{F_2}, \quad \text{with} \quad W=F_1^2-F_0F_2. \tag{3}$$

The factor sign (F_0) is deduced from the conditions $W>0$ and $e^{md}>0$. The next step is the application of the *regula falsi* to the points (x_1,H_1) and (x_2,H_2), which leads to the expression

$$x_3=\frac{x_1H_2-x_2H_1}{H_2-H_1}=x_1-\frac{d}{H_2/H_1-1} \tag{4}$$

where x_3 is the first approximation of the root of $F(x)$ and $H_2/H_1=F_2\cdot e^{md}/F_1$. Equation (4) can be written in the form

$$x_3=x_1+\text{sign}(F_0)\cdot\frac{F_1\cdot d}{\sqrt{W}}. \tag{5}$$

To avoid the factor sign (F_0) we divide numerator and denominator by F_0 and obtain the final expression for the algorithm:

$$x_3=x_1+d\cdot\frac{F_1/F_0}{\sqrt{(F_1/F_0)^2-F_2/F_0}}. \tag{6}$$

When $F_0\cdot F_2<0$, x_3 will be on the interval $[x_0,x_2]$ so convergence is guaranteed.

After computation of the first iterate x_3 we build up a new interval consisting of x_3 and one of the other remaining x values in such a way that $F_3\cdot F_n<0$ $(n=0,1,2)$ in order to be sure that the next iterate will remain on the starting interval. The procedure is depicted in Fig. 1.

The described method can even be used when $F_0=F_1$ or $F_1=F_2$ as can accidentally happen.

Suppose $F(x)=x^3-x-5$ and we choose $[-1,3]$ as the starting interval.

$$F_0=F_1=-5; \qquad F_2=19.$$

For x_3 we compute the value 1.9128, which is already fairly close to the root $1.904160859\cdots$.

Manuscript received January 31, 1979.

The author is with the Department of Electrical Engineering, Delft University of Technology, Delft, The Netherlands.

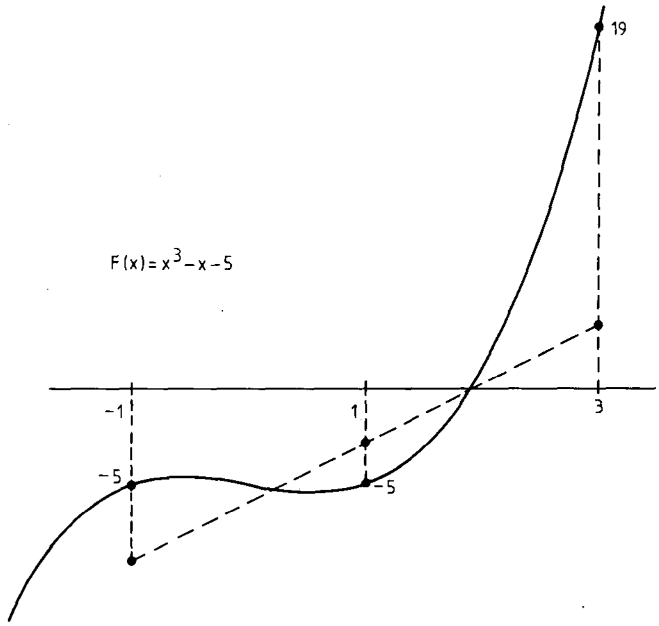

Fig. 2.

As $F_3 > 0$ we decide to take $[x_1, x_3]$ as the next interval of computation (Fig. 2).

The procedure can be terminated when a given accuracy is obtained.

II. Rate of Convergence

Let $e_n = x_n - r$ be the actual error between x_n and the root r. By means of a Taylor expansion in the vicinity of r we get $F_n \sim e_n f + e_n^2 g + e_n^3 h$ with $f = F'(r)$, $g = \frac{1}{2}F''(r)$, and $h = \frac{1}{6}F'''(r)$ $\cdot d = e_1 - e_0 = e_2 - e_1$ so the error at the first iterate is

$$e_3 = e_1 - e_1 \cdot \frac{(e_1 - e_0)(f + e_1 g + e_1^2 h)}{\sqrt{W}} \tag{7}$$

which can be derived from (5). This expression is valid for all possible shapes of $F(x)$.

$$W = e_1^2 (f + e_1 g + e_1^2 h)^2 - e_0 e_2 (f + e_0 g + e_0^2 h)(f + e_2 g + e_2^2 h).$$

After some adequate approximations we get

$$W \sim (e_1 - e_0)^2 \cdot \left[f^2 + g^2 (e_1^2 + 2 e_0 e_1 - e_0^2) + 2 e_1 f g + 2 f h (e_1 - e_0)^2 \right].$$

When $F_1 \to 0$, $e_1 \to 0$, and $e_1^2 \ll |e_0 e_2|$ so

$$e_3 \sim \frac{1}{2} e_0 e_1 e_2 \frac{g^2 - 2fh}{f^2}. \tag{8}$$

III. Examples

$F(x) = xe^x - 10$ on $[-10, 10]$

$x_3 = 0.06\cdots$
$2.75\cdots$
$1.71\cdots$
$1.746\cdots$
$1.74552798\cdots$
1.745528003

on $[-100, 100]$

$x_3 = 6.10^{-21}$
$7.74\cdots$
$2.38\cdots$
$1.709\cdots$
$1.7458\cdots$
$1.745527990\cdots$
1.745528003

$F(x) = (\tan x)^{\tan x} - 10^3$ on $[1.3, 1.4]$

$x_3 = 1.352\cdots$
$1.356\cdots$
$1.3547099\cdots$
1.354710442

on $[0, 1.5]$

$x_3 = 0.75\cdots$
$1.12\cdots$
$1.31\cdots$
$1.40\cdots$
$1.357\cdots$
$1.35429\cdots$
1.354710756
$1.354710442.$

$F(x) = \sin x$ on $[10, 280]$, x in degrees. A trivial example.

$x_3 = 254.50\cdots$
$177.09\cdots$
$179.97\cdots$
$179.99995\cdots$
$180.$

IV. Conclusion

The proposed algorithm offers a good rate of convergence and is suitable especially on those cases where $F(x)$ is not strictly monotone. The method can be used when other three-point iterative methods (e.g., exponential or hyperbolic) fail.

Monotonic Magnitude Response with Equal Ripple Sensitivity

D. M. RABRENOVIĆ AND Ž. J. ALEKSIĆ

***Abstract*—Sensitivity of all pole filters with critical monotonic (CM) magnitude characteristic is optimized in a mini-max sense, and a new class of transitional monotonic filters is introduced. Comparison with other monotonic filters is also given.**

Although different aspects of circuit sensitivity have been extensively treated in the literature, much attention has not been paid in the existing literature to the problem of sensitivity minimization by an appropriate choice of the magnitude characteristic.

In order to concentrate on the shape of the magnitude characteristic we shall consider the well-known expression for the summed sensitivity of the magnitude response of an RC active network with tracking components [1]. Equation (1) relates the variation of the logarithmic gain Δa to the slope of the magnitude response and the relative tolerances of the passive elements $\Delta R/R$ and $\Delta C/C$:

$$\Delta a = \left(\frac{\Delta R}{R} + \frac{\Delta C}{C} \right) \frac{\omega}{|A|} \frac{d|A|}{d\omega} \tag{1}$$

which for $\Delta R/R = \Delta C/C$ reduces to

$$\Delta a = 2 \frac{\Delta R}{R} \frac{\omega}{|A|} \frac{d|A|}{d\omega} = kD \tag{2}$$

Manuscript received August 1, 1978; revised December 27, 1978.
The authors are with the Faculty of Electrical Engineering, University of Belgrade, 11001 Belgrade, Yugoslavia.

0098-4094/79/1100-0980$00.75 ©1979 IEEE